

UNIQUENESS OF CONSTANT SCALAR CURVATURE SASAKIAN METRICS

XISHEN JIN AND XI ZHANG

ABSTRACT. In this paper, we prove that the transverse Mabuchi K -energy functional is convex along the weak geodesic in the space of Sasakian metrics. As an application, we obtain the uniqueness of constant scalar curvature Sasakian metrics modulo automorphisms for the transverse holomorphic structure.

1. INTRODUCTION

Let (M, g) be a connected oriented $2m + 1$ -dimensional Riemannian manifold. If the cone manifold $(C(M), \tilde{g}) = (M \times R^+, r^2g + dr^2)$ is Kähler, we say (M, g) is a Sasakian manifold. It is well known that (M, g) is a Sasakian-Einstein manifold if and only if the Kähler cone $(C(M), \tilde{g})$ is a Calabi-Yau cone. Sasakian geometry was introduced by Sasaki [21] fifty years ago, and it can be seen as an odd-dimensional counterpart of Kähler geometry. Since Sasakian geometry has been proved to be a rich source for the production of positive Einstein metrics [6, 11], and the existence of Sasakian-Einstein metrics is of great interest in the physics of the famous Ads/CFT duality conjecture [16, 17, 18, 19], there has been renewed interest in Sasakian manifolds recently.

The aim of this paper is to study the uniqueness of Sasakian metrics with constant scalar curvature on compact Sasakian manifolds. The uniqueness of Sasakian-Einstein metrics was proved by Cho, Futaki and Ono ([9]) for the toric case, and by Nitta and Sekiya ([20]) for the general case. In [14], Guan and the second author studied the geodesic equation in the space of Sasakian metrics on a compact Sasakian manifold (M, g) , and obtained the weak C^2 regularity of such geodesic. Then, they proved that the constant scalar curvature Sasakian metric (cscS metric) is unique in each basic Kähler class if the first basic Chern class is either strictly negative or zero.

In [1], Berman and Berndtsson proved the convexity of the Mabuchi K -energy along the weak geodesic in the space of Kähler metrics by using Chen's weak C^2 regularity ([3, 4, 8]). As an application, they obtained the uniqueness of constant scalar curvature Kähler metric (cscK metric) modulo automorphisms on a compact Kähler manifold in a fixed Kähler class.

A Sasakian manifold (M, g) has one dimensional foliation \mathcal{F}_ξ associated to the characteristic Reeb vector field ξ , which admits a transverse holomorphic structure. Another Sasakian metric g' is compatible with g means that they have the same

AMS Mathematics Subject Classification. 53C55, 32W20.

The authors were supported in part by NSF in China No.11131007, 11571332 and the Hundred Talents Program of CAS.

Reeb vector field, the same transverse holomorphic structure and the same holomorphic structure on the cone $C(M)$, see section 2 for details. In this paper, by using the weak C^2 regularity in [14] and following the argument of Berman and Berndtsson ([1]), we prove the uniqueness of constant scalar curvature Sasakian metrics (cscS) up to the action of the identity component of the automorphism group for the transverse holomorphic structure, which is denoted to be G_0 in Definition 4.2. In fact, we obtain the following theorem.

Theorem 1.1. *Let (M, g) be a compact Sasakian manifold, $(\xi, \eta_1, \Phi_1, g_1)$ and $(\xi, \eta_2, \Phi_2, g_2)$ are two constant scalar curvature Sasakian metrics compatible with g . Then there exists ι in the group G_0 , such that $d\eta_2 = \iota^* d\eta_1 = d\eta_1 + d_B d_B^c \varphi_\iota$. Furthermore, for the two Sasakian structures, we have the following relations*

$$(1.1) \quad \begin{aligned} \eta_2 &= \eta_1 + d_B^c \varphi_\iota, \\ \Phi_2 &= \Phi_1 - \xi \otimes d_B^c \varphi_\iota \circ \Phi, \\ g_2 &= \frac{1}{2} d\eta_2 \circ (Id \otimes \Phi_2) + \eta_2 \otimes \eta_2. \end{aligned}$$

This paper is organized as follow. In Section 2, we will recall some preliminary results about Sasakian geometry, in particular, the weak geodesic established in [14]. In Section 3, we prove the convexity of the transverse Mabuchi K -energy \mathcal{M} along the weak geodesic. In Section 4, we give a proof of Theorem 1.1, as an application of the convexity of \mathcal{M} .

2. PRELIMINARY RESULTS IN SASAKIAN GEOMETRY

There are many structures on Sasakian manifold. A Sasakian manifold (M, g) has a contact structure (ξ, η, Φ) , and it also has a one-dimensional foliation \mathcal{F}_ξ , called the Reeb foliation, which has a transverse Kähler structure. Here, the killing vector field ξ is called the characteristic or Reeb vector field, η is called the contact 1-form, and Φ is an $(1, 1)$ -tensor field which defines a complex structure on the contact sub-bundle $\mathcal{D} = \ker \eta$. In the following, a Sasakian manifold will be denoted by (M, ξ, η, Φ, g) , and the quadruple (ξ, η, Φ, g) will be called by a Sasakian structure on a manifold M .

Let (M, ξ, η, Φ, g) be a $(2n+1)$ -dimensional Sasakian manifold, and let \mathcal{F}_ξ be the characteristic foliation generated by ξ . A transverse holomorphic structure on \mathcal{F}_ξ is given by an open covering $\{U_i\}_{i \in A}$ of M and local submersion of $f_i : U_i \rightarrow \mathbb{C}^m$ with fibers of dimension 1, such that for $i, j \in A$, there is a holomorphic isomorphism θ_{ij} of open sets of \mathbb{C}^m such that $f_i = \theta_{ij} \circ f_j$ on $U_i \cap U_j$.

Now, we consider the quotient bundle of the foliation \mathcal{F}_ξ , $\nu(\mathcal{F}_\xi) = TM/L\xi$. The metric g gives a bundle isomorphism σ between $\nu(\mathcal{F}_\xi)$ and the contact sub-bundle \mathcal{D} , where $\sigma : \nu(\mathcal{F}_\xi) \rightarrow \mathcal{D}$ is defined by

$$\sigma([X]) = X - \eta(X)\xi.$$

By this isomorphism, $\Phi|_{\mathcal{D}}$ induces a complex structure \bar{J} on $\nu(\mathcal{F}_\xi)$. $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$ gives M a transverse Kähler structure with transverse Kähler form $\frac{1}{2}d\eta$ and metric g^T defined by $g^T = \frac{1}{2}d\eta(\cdot, \Phi\cdot)$. For the transverse metric g^T , one can define the transverse Levi-Civita connection ∇^T on \mathcal{D} by

$$\nabla_X^T Y = \begin{cases} (\nabla_X Y)^p, & X \in \mathcal{D} \\ [\xi, Y]^p, & X = \xi \end{cases}$$

2

where Y is a section of \mathcal{D} and X^p the projection of X onto \mathcal{D} . One can check that the transverse Levi-Civita connection is torsion-free and metric compatible. The transverse curvature relating to the above transverse connection will be denoted by $R^T(V, W)Z$, where $V, W \in TM$ and $Z \in \mathcal{D}$. We define the transverse Ricci curvature by

$$\text{Ric}^T(X, Y) = \langle R^T(X, e_i)e_i, Y \rangle_g,$$

where e_i is the orthonormal basis of \mathcal{D} and $X, Y \in \mathcal{D}$. Furthermore, $\rho^T = \text{Ric}^T(\Phi \cdot, \cdot)$ is called the transverse Ricci form analog to the Kähler geometry. The transverse scalar curvature S^T is defined to be the trace of ρ^T with respect to g^T . According to standard computation, we have that $S^T = S + 2n$, where S is the scalar curvature of g in the usual sense.

We say a p -form θ is basic, if it satisfies that

$$i_\xi \theta = 0, \text{ and } L_\xi \theta = 0.$$

It is easy to check that $d\theta$ is also basic, if θ is, i.e. the exterior differential preserves basic forms. Let $\wedge_B^p(M)$ be the sheaf of germs of basic p -forms and $\Omega_B^p(M) = \Gamma(M, \wedge_B^p(M))$ be the set of all sections of $\wedge_B^p(M)$. The basic cohomology can be defined in a usual way([15]). Let $\mathcal{D}^{\mathbb{C}}$ be the complexification of the sub-bundle \mathcal{D} , and we can decompose it into its eigenspaces with respect to $\Phi|_{\mathcal{D}}$, that is

$$\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}.$$

Similarly, we have a splitting of the complexification of the bundle $\wedge_B^p(M)$ of basic p -forms on M ,

$$\wedge_B^p(M) \otimes \mathbb{C} = \bigoplus_{i+j=p} \wedge_B^{i,j}(M),$$

where $\wedge_B^{i,j}(M)$ denotes the bundle of basic forms of type (i, j) . Define ∂_B and $\bar{\partial}_B$ by

$$\begin{aligned} \partial_B : \wedge_B^{i,j}(M) &\rightarrow \wedge_B^{i,j+1}(M), \\ \bar{\partial}_B : \wedge_B^{i,j}(M) &\rightarrow \wedge_B^{i+1,j}(M), \end{aligned}$$

which is the decomposition of d . Let $d_B^c = \frac{1}{2}\sqrt{-1}(\bar{\partial}_B - \partial_B)$, and $d_B = d|_{\wedge_B^p}$. We have $d_B = \partial_B + \bar{\partial}_B$ and $d_B d_B^c = \sqrt{-1}\partial_B \bar{\partial}_B$, and $d_B^2 = (d_B^c)^2 = 0$. The basic cohomology groups $H_B^{i,j}(M, \mathcal{F}_\xi)$ are fundamental invariants of a Sasakian structure which enjoy many of the same properties as the Dolbeault cohomology of a Kähler structure. On Sasakian manifolds, the $\partial\bar{\partial}$ -lemma holds for basic forms.

Proposition 2.1 ([15]). *Let θ and θ' be two real closed basic forms of type $(1, 1)$ on a compact Sasakian manifold (M, ξ, η, Φ, g) . If $[\theta]_B = [\theta']_B \in H_B^{1,1}(M, \mathcal{F}_\xi)$, then there is a real-valued basic function φ such that*

$$\theta = \theta' + \sqrt{-1}\partial_B \bar{\partial}_B \varphi.$$

Now we consider the deformation of the Sasakian structures. Let us denote the space of all smooth basic functions φ , i.e. $\xi\varphi = 0$ on (M, ξ, η, Φ, g) by $C_B^\infty(M, \xi)$. And specially

$$\mathcal{H}(\xi, \eta, \Phi, g) = \{\varphi \in C_B^\infty(M, \xi) : \eta_\varphi \wedge (d\eta_\varphi)^n \neq 0\},$$

where $\eta_\varphi = \eta + d_B^c \varphi$, and $d\eta_\varphi = d\eta + d_B d_B^c \varphi$. The space $\mathcal{H}(\xi, \eta, \Phi, g)$ is contractible, and we will denote it by \mathcal{H} for simplicity. For $\varphi \in \mathcal{H}$, $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ is also a Sasakian structure on M , where

$$\Phi_\varphi = \Phi - \xi \otimes (d_B^c \varphi) \circ \Phi, \text{ and } g_\varphi = \frac{1}{2} d\eta_\varphi \circ (Id \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi.$$

As in [13, 14], we define a functional $\mathcal{I} : \mathcal{H} \rightarrow \mathbb{R}$ by

$$(2.1) \quad \mathcal{I}(\varphi) = \sum_{p=0}^n \frac{n!}{(p+1)!(n-p)!} \int_M \varphi (d\eta)^{n-p} \wedge (\sqrt{-1} \partial_B \bar{\partial}_B \varphi)^p \wedge \eta.$$

Set

$$\mathcal{H}_0 = \{\varphi \in \mathcal{H} | \mathcal{I}(\varphi) = 0\},$$

and

$$\mathcal{K} = \{\text{transverse Kähler form in the basic } (1, 1) \text{ class } [d\eta]_B\}.$$

Then we have $\mathcal{H}_0 \cong \mathcal{K}$. In [13], Guan and the second author proved that \mathcal{H}_0 is totally geodesic and totally convex in \mathcal{H} . And the tangent space of \mathcal{H}_0 at φ is the set

$$(2.2) \quad T\mathcal{H}_0|_\varphi = \{\psi \in \mathcal{H} | \int_M \psi (d\eta_\varphi)^n \wedge \eta = 0\}.$$

It is well known that both $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ and (ξ, η, Φ, g) have the same transverse holomorphic structure on $\nu(\mathcal{F}_\xi)$ and the same holomorphic structure on the cone $C(M)$ ([5, 10]). Obviously, these deformations of Sasakian structure deform the transverse Kähler form in the same basic $(1, 1)$ class. As in [7], we call this class the basic Kähler class of the Sasakian manifold (M, ξ, η, Φ, g) . It should be noted that the contact bundle \mathcal{D} may change under such deformations. We define $\mathcal{S}(\xi, \bar{J})$ to be the subset of all structures $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ in $\mathcal{S}(\xi)$ such that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{\tilde{\Phi}} & TM \\ \pi_\nu \downarrow & & \downarrow \pi_\nu \\ \nu(\mathcal{F}_\xi) & \xrightarrow{\bar{J}} & \nu(\mathcal{F}_\xi) \end{array}$$

commutes, where $\mathcal{S}(\xi)$ denotes all Sasakian structure $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$, such that $\tilde{\xi} = \xi$. The set $\mathcal{S}(\xi, \bar{J})$ consists of elements of $\mathcal{S}(\xi)$ with the same transverse holomorphic structure \bar{J} . For two different Sasakian structure in $\mathcal{S}(\xi, \bar{J})$, we have ([5, 7, 22]):

Lemma 2.2. *If (ξ, η, Φ, g) and $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ are two Sasakian structures in $\mathcal{S}(\xi, \bar{J})$, then there exist real-valued basic functions φ and ψ and integral closed 1-form α , such that*

$$\begin{aligned} \tilde{\eta} &= \eta + d_B^c \varphi + d\psi + i(\alpha), \\ \tilde{\Phi} &= \Phi - \xi \otimes (\tilde{\eta} - \eta) \circ \Phi, \\ \tilde{g} &= \frac{1}{2} d\tilde{\eta} \circ (Id \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}, \end{aligned}$$

where $d\tilde{\eta} = d\eta + d_B d_B^c \varphi$. In particular, if this two Sasakian structures induce the same holomorphic structure on the cone $C(M)$, we have that $\tilde{\eta} = \eta + d_B^c \varphi$.

Definition 2.1. Given any Sasakian manifold (M, ξ, η, Φ, g) , we say another Sasakian metric g' is compatible with the Sasakian structure of (M, ξ, η, Φ, g) , if they have the same Reeb vector field, the same transverse holomorphic structure $\nu(\mathcal{F}_\xi)$ and the same holomorphic structure on the cone $C(M)$.

By Lemma 2.2., if Sasakian structure (ξ, η', Φ', g') is compatible with (ξ, η, Φ, g) , then there must exists a basic function in $\varphi \in \mathcal{H}$ such that $\eta' = \eta + d_B^c \varphi$. In the following, we also say φ the transverse Kähler potential of g' . Similar to the Kähler case, we know that the average

$$\bar{S}^T = \frac{\int_M S_\varphi^T (d\eta_\varphi)^n \wedge \eta_\varphi}{\int_M (d\eta_\varphi)^n \wedge \eta_\varphi} = \frac{\int_M 2n\rho_\varphi^T \wedge (d\eta_\varphi)^{n-1} \wedge \eta_\varphi}{\int_M (d\eta_\varphi)^n \wedge \eta_\varphi}$$

is a constant independent of the choice of $\varphi \in \mathcal{H}$. We say $g_\varphi \in \mathcal{K}$ (or $\varphi \in \mathcal{H}$) is a constant scalar curvature Sasakian (cscS) metric, if $S_\varphi = \bar{S}^T + 2n$, which is equivalent to $S_\varphi^T = \bar{S}^T$. Indeed, this equation is an elliptic equation of forth order. As in the Kähler case, the Mabuchi K-energy is a useful tool to consider such metrics. In the Sasakian case, the Mabuchi K-energy is defined in [10], and we recall it in the following lemma.

Lemma 2.3. *Let φ_1 and φ_2 are two basic functions in \mathcal{H} and φ_t ($t \in [a, b]$) be a path in \mathcal{H} connecting φ_1 and φ_2 . Then*

$$\begin{aligned} \mathcal{M}(\varphi_1, \varphi_2) &= - \int_a^b \int_M \dot{\varphi}_t (S_{\varphi_t}^T - \bar{S}^T) (d\eta_{\varphi_t})^n \wedge \eta_{\varphi_t} \wedge dt \\ &= - \int_a^b \int_M \dot{\varphi}_t (S_{\varphi_t}^T - \bar{S}^T) (d\eta_{\varphi_t})^n \wedge \eta \wedge dt \end{aligned}$$

is independent of the path φ_t , where $\dot{\varphi}_t = \frac{d\varphi}{dt}$. Furthermore, \mathcal{M} satisfies the 1-cocycle condition

$$(2.3) \quad \mathcal{M}(\varphi_1, \varphi_2) + \mathcal{M}(\varphi_2, \varphi_3) = \mathcal{M}(\varphi_1, \varphi_3)$$

and

$$(2.4) \quad \mathcal{M}(\varphi_1 + C_1, \varphi_2 + C_2) = \mathcal{M}(\varphi_1, \varphi_2)$$

for any $C_1, C_2 \in \mathbb{R}$.

In view of (2.4), $\mathcal{M} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ factors through $\mathcal{H}_0 \times \mathcal{H}_0$. Hence we can define the mapping $\mathcal{M} : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{R}$ by the identity $\mathcal{K} \cong \mathcal{H}_0$,

$$\mathcal{M}(d\eta_{\varphi_1}, d\eta_{\varphi_2}) := \mathcal{M}(\varphi_1, \varphi_2).$$

Definition 2.2. The mapping $\mu : \mathcal{K} \rightarrow \mathbb{R}$, where $\mu(d\eta_\varphi)$ is defined by

$$\mathcal{M}(d\eta_\varphi) = \mathcal{M}(d\eta, d\eta_\varphi)$$

is called the K-energy map of the transverse Kähler class in $[d\eta]_B$. The mapping $\mathcal{M} : \mathcal{H} \rightarrow \mathbb{R}$, $\mathcal{M}(\varphi) = \mathcal{M}(0, \varphi)$ is also called by the K-energy map of the space \mathcal{H} .

It is easy to see that cscS metric is a critical point of \mathcal{M} . In order to consider the uniqueness of such cscS metrics in the space \mathcal{H} , we will consider the convexity of the K-energy along the geodesics in \mathcal{H} .

In [14], Guan and the second author studied the geodesic equation in \mathcal{H} . Here, we recall some results. The Weil-Peterson metric in the space \mathcal{H} is defined as

$$(\psi_1, \psi_2)_\varphi = \int_M \psi_1 \psi_2 (d\eta_\varphi)^n \wedge \eta, \quad \forall \psi_1, \psi_2 \in T\mathcal{H}.$$

A nature connection of the metric can be defined to be

$$(2.5) \quad D_{\dot{\varphi}}\psi = \dot{\psi} - \frac{1}{4} \langle d_B \dot{\varphi}, d_B \psi \rangle_{g_\varphi}$$

where $\psi \in C_B^\infty(M, \xi) = T\mathcal{H}$. And in particular, if $\varphi_t(t \in [a, b])$ is a geodesic, then it satisfies that

$$(2.6) \quad \frac{\partial^2 \varphi_t}{\partial t^2} - \frac{1}{4} |d_B \frac{\partial \varphi_t}{\partial t}|_{g_\varphi}^2 = 0.$$

In [14], Guan and the second author reduced the geodesic equation (2.6) to the Dirichlet problem of a degenerate Monge-Ampère type equation on the Kähler cone $C(M) = M \times \mathbb{R}^+$. And for convenience, we recall the key observation. They denoted a new function ψ on $M \times [1, \frac{3}{2}] \subset C(M)$ by converting the time variable t to the radial variable r as follow,

$$(2.7) \quad \psi(\cdot, r) = \varphi_{2(r-1)}(\cdot) + 4 \log r.$$

By setting a $(1, 1)$ -form on $M \times [1, \frac{3}{2}]$ such that

$$(2.8) \quad \Omega_\psi = \tilde{\omega} + \frac{r^2}{2} \sqrt{-1} (\partial \bar{\partial} \psi - \frac{\partial \psi}{\partial r} \partial \bar{\partial} r),$$

where $\tilde{\omega} = \frac{1}{2} dd^c r^2$ is the fundamental form of the Kähler cone, they concluded that the geodesic equation (2.6) is equivalent to the following degenerate Monge-Ampère type equation

$$(2.9) \quad (\Omega_\psi)^{n+1} = 0, \text{ on } M \times [1, \frac{3}{2}].$$

In this paper, we give another description of the geodesic equation similar to the Kähler case. And we will reduce it to a degenerate transverse Monge-Ampère type equation on $M \times D$, where D is an annulus in \mathbb{C} . We let $t = \log |\tau|$, and $\Psi(\cdot, \tau) = \varphi_t(\cdot)$, i.e. Ψ is a radial function defined on $M \times D$, where $D = \{z \in \mathbb{C} | a \leq \log z \leq b\}$.

Proposition 2.4. *The geodesic equation (2.6) or (2.9) can be written as the following transverse Monge-Ampère equation on $X = M \times D$ for the function Ψ ,*

$$(2.10) \quad (\pi^* d\eta + \tilde{d} \tilde{d}^c \Psi)^{n+1} \wedge \eta = 0,$$

where π is the projection from X to M and

$$\tilde{d} = d_B + d_\tau \text{ and } \tilde{d}^c = d_B^c + d_\tau^c, \quad d_\tau^c = \frac{\sqrt{1}}{2} (\bar{\partial}_\tau - \partial_\tau).$$

Proof. Direct computation implies that

$$(2.11) \quad \begin{aligned} & (\pi^* d\eta + \tilde{d} \tilde{d}^c \Psi)^{n+1} \wedge \eta \\ &= \frac{n+1}{4 |\tau|^2} \left(\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4} |d_B \frac{\partial \varphi}{\partial t}|_{g_\varphi}^2 \right) \sqrt{-1} \partial \tau \wedge \bar{\partial} \tau \wedge (\pi^* d\eta_\varphi)^n \wedge \eta. \end{aligned}$$

According to the computation of [14], we know that

$$(2.12) \quad (\Omega_\psi)^{n+1} = (n+1) 2^{-n} r^{2n+3} \left(\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4} |d_B \frac{\partial \varphi}{\partial t}|_{g_\varphi}^2 \right) dr \wedge \eta \wedge (d\eta_\varphi)^n,$$

and

$$(2.13) \quad \tilde{\omega}^{n+1} = (n+1)2^{-n}r^{2n+3}dr \wedge \eta \wedge (d\eta)^n.$$

Furthermore, we have that

$$(2.14) \quad \frac{(\pi^*d\eta + \tilde{d}\tilde{d}^c\Psi)^{n+1} \wedge \eta}{\frac{n+1}{4|\tau|^2}\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau \wedge (\pi^*d\eta)^n \wedge \eta} = \frac{(\Omega_\psi)^{n+1}}{\tilde{\omega}^{n+1}},$$

which implies the result required. \square

The equation (2.6) usually does not admit a smooth solution, i.e. we can not always find smooth geodesic in \mathcal{H} . But according to [14] (Theorem 1.), there exists a unique weak C^2 solution Ψ , i.e. we have the following lemma.

Lemma 2.5. *For any two functions $\varphi_a, \varphi_b \in \mathcal{H}$, there exists a unique geodesic path Ψ connecting them, such that $\Psi|_{M \times a} = \varphi_a$, and $\Psi|_{M \times b} = \varphi_b$. In particular,*

$$\|\Psi\|_{C^1(M \times D)} + \sup_{M \times D} |\Delta\Psi| \leq C,$$

where C is a constant depending only on (ξ, η, Φ, g) , $\|\varphi_a\|_{C^{2,1}}$ and $\|\varphi_b\|_{C^{2,1}}$. Furthermore, $\Psi(\cdot, t)$ is basic and $\pi^*d\eta + \tilde{d}\tilde{d}^c\Psi \geq 0$ in the sense of L^∞ on $M \times D$.

Remark 2.1. Indeed, we use the fact that the C^1 -norm and $\Delta\Psi$ can be controlled by the function $\psi(\cdot, r)$ in [14], since r is away from 0.

3. CONVEXITY OF K-ENERGY ALONG WEAK GEODESIC

Since our original definition of the Mabuchi K-energy depends on the forth order derivative, we want to rewrite an explicit formula for it, which has an “energy part” and “entropy part” as in the Kähler case(see [1] for the Kähler case). We begin with some notations. Given $\varphi \in \mathcal{H}$, we will write the energy as follow

$$\mathcal{E}(\varphi) = \sum_{j=0}^n \int_M \varphi(d\eta_\varphi)^{n-j} \wedge (d\eta)^j \wedge \eta,$$

and

$$\mathcal{E}^T(\varphi) = \sum_{j=0}^{n-1} \int_M \varphi(d\eta_\varphi)^{n-j-1} \wedge (d\eta)^j \wedge T \wedge \eta,$$

where T is a basic form of $(1, 1)$ -type. It is easy to check that

$$(3.1) \quad d\mathcal{E}|_\varphi = (n+1)(d\eta_\varphi)^n \wedge \eta, \text{ and } d\mathcal{E}^T|_\varphi = n(d\eta_\varphi)^{n-1} \wedge T \wedge \eta.$$

Remark 3.1. By writing $d\mathcal{E}|_\varphi = \Xi$, where Ξ is a measure on M , we mean that

$$\left. \frac{d\mathcal{E}(\varphi_t)}{dt} \right|_{t=0} = \int_M \dot{\varphi}_t|_{t=0} \Xi,$$

where $\{\varphi_t\}$ is a curve in \mathcal{H} such that $\varphi_t|_{t=0} = \varphi$.

Similarly, the second order variations of \mathcal{E} and \mathcal{E}^T are

$$\begin{aligned} d_\tau d_\tau^c \mathcal{E}(\varphi_t) &= \int_M (\pi^*d\eta + \tilde{d}\tilde{d}^c\Psi)^{n+1} \wedge \eta, \\ d_\tau d_\tau^c \mathcal{E}^T(\varphi_t) &= \int_M (\pi^*d\eta + \tilde{d}\tilde{d}^c\Psi)^n \wedge \pi^*T \wedge \eta. \end{aligned}$$

Finally, we consider the entropy of a measure μ relative to a reference measure μ_0 is defined as follows, if μ is absolutely continuous with respect to μ_0 , then

$$H_{\mu_0}(\mu) := \int_M \log \frac{\mu}{\mu_0}.$$

We have the following proposition for the entropy defined above(see [1]).

Proposition 3.1. *If μ_0 and μ are probability measures on M such that μ is absolutely continuous with respect to μ_0 , then*

$$H_{\mu_0}(\mu) = \sup_f \left(\int_M f \mu - \log \int_M e^f \mu_0 \right)$$

where the supremum is taken over all continuous functions on M . Furthermore, it is a convex function of the measure μ for the natural affine structure on the space of probability measures, and lower semicontinuous with respect to the weak $*$ -topology.

We can give the explicit formula of Mabuchi K-energy in terms of the energy and entropy defined above.

Proposition 3.2. *With the notation $\mu_0 = (d\eta)^n \wedge \eta$, the following formula holds for the Mabuchi K-energy on \mathcal{H} :*

$$\mathcal{M}(\varphi) = \frac{\bar{S}^T}{n+1} \mathcal{E}(\varphi) - 2\mathcal{E}^{\rho_{d\eta}}(\varphi) + 2H_{\mu_0}((d\eta_\varphi)^n \wedge \eta).$$

Remark 3.2. This proposition can be proved by checking the first derivative of both side along a fixed path. And they are both equal to 0, while $\varphi = 0$. Furthermore, the new formula of Mabuchi K-energy is well-defined for any basic function with weak C^2 -regularity, i.e. functions φ such that $d\eta + d_B d_B^c \varphi$ has L^∞ -coefficients.

Now we consider the convexity of the Mabuchi K-energy along the weak geodesics, modifying the method of [1].

Theorem 3.3. *Let $\varphi_t(\cdot) = \varphi_\tau(\cdot) = \Psi(\cdot, \tau)$ be a family of basic functions, such that $d\eta + d_B d_B^c \varphi_\tau$ has L^∞ -coefficients, $\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi \geq 0$, and $(\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^{n+1} \wedge \eta = 0$ on $M \times D$. Then the Mabuchi K-energy $\mathcal{M}(\varphi_\tau)$ is weakly subharmonic with respect to $\tau \in D$. In particular, $\mathcal{M}(\varphi_t)$ is weakly convex along the weak geodesic φ_t connecting two given points in \mathcal{H} .*

Proof. Similar to [1], we can prove that

$$d_\tau d_\tau^c \mathcal{M}^u(\tau) = \int_M \tilde{d}\tilde{d}^c u \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta,$$

where u is a locally bounded function on $M \times D$ and

$$\mathcal{M}^u(\tau) := \frac{\bar{S}^T}{n+1} \mathcal{E}(\varphi_\tau) - 2\mathcal{E}^{\rho_{d\eta}}(\varphi_\tau) + 2 \int_M u(\cdot, \tau) (d\eta_{\varphi_\tau})^n \wedge \eta.$$

We want to apply the above considerations to $u = \log \frac{((d\eta + d_B d_B^c \Psi)^n \wedge \eta)}{(d\eta)^n \wedge \eta}$, but this functions is not locally bounded. We will introduce a truncation in the following way as in [1]. For a fixed positive number A , we define

$$\Psi_A := \max\left\{\log \frac{(d\eta + d_B d_B^c \Psi)^n}{(d\eta)^n \wedge \eta}, \chi - A\right\}$$

where χ denotes to be $\chi = \pi^* \chi_0 - \Psi$ for $\chi_0 \in \mathcal{H}$, such that $d\eta_{\chi_0} = d\eta + d_B d_B^c \chi_0 \geq 0$. So we have that

$$\tilde{d}\tilde{d}^c \chi = \pi^* d_B d_B^c \chi_0 - (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi) + \pi^* d\eta \geq -(\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi).$$

We claim that for each $A > 0$, $T_A = \int_M \tilde{d}\tilde{d}^c \Psi_A \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta$ is defined by a nonnegative measure on D , i.e. for any nonnegative compact supported smooth function ϱ ,

$$(3.2) \quad T_A(\varrho) = \int_D \varrho T_A \geq 0.$$

In order to prove this claim, we will consider the localization of Sasakian manifold.

In [12], it has been proved that every Sasakian metric can be locally generated by a real function of $2n$ variables, i.e. the Sasakian analog of the Kähler potential for the Kähler geometry. This local property has also been applied in [23] to prove the existence of α -invariant in Sasakian case. Let ζ_j be a partition of unity subordinate to a covering of coordinate patches such that the supported set of ζ_j is a subset of $\Omega_{p_j}^{-1}([-\frac{R}{2}, \frac{R}{2}] \times B_o(\frac{R}{2}))$ for a local coordinate chart (U_{p_j}, Ω_{p_j}) such that $\Omega_{p_j}(p_j) = 0$, and for convenience we write

$$U_{p_j} = \Omega_{p_j}^{-1}([-\frac{R}{2}, \frac{R}{2}] \times B_o(\frac{R}{2})).$$

Furthermore, we choose R is sufficiently small such that we can write

$$(3.3) \quad d\eta = d_B d_B^c \rho_j$$

on U_{p_j} , for some real-valued basic function ρ_j on M . The compactness of M implies that we can choose such $R > 0$ for all $p \in M$. For such partition of unity, we write $T_A = \sum_j T_A^j$, where

$$(3.4) \quad \begin{aligned} T_A^j &= \int_M \zeta_j \tilde{d}\tilde{d}^c \Psi_A \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta \\ &= \int_{U_{p_j}} \zeta_j \tilde{d}\tilde{d}^c \Psi_A \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta. \end{aligned}$$

Since $\rho_j \circ \pi(\tau, \cdot) + \varphi_\tau$ defines a plurisubharmonic(psh) function on $B_o(\frac{R}{2}) \subset \mathbb{C}^m$, we know that we can approximate the measure $(d\eta + d_B d_B^c \Psi)^n$ by the Bergman measure $\beta_{j,k} = \beta_{k(\rho_j \circ \pi(\tau, \cdot) + \varphi_\tau)}$ for the Hilbert space of all holomorphic functions on $B_o(\frac{R}{2})$ with the weight $k(\rho_j \circ \pi(\tau, \cdot) + \varphi_\tau)$, according to [1]. More precisely, we consider the following measure on D ,

$$(3.5) \quad T_{A,k}^j = \int_{U_{p_j}} \zeta_j \tilde{d}\tilde{d}^c \Psi_{A,k} \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta,$$

where $\Psi_{A,k} = \max\{\log \beta_{j,k}, \chi - A\}$. By the results on plurisubharmonic variation of Bergman kernels in [2], we know that

$$(3.6) \quad \tilde{d}\tilde{d}^c \log \beta_{j,k} \geq -k \tilde{d}\tilde{d}^c (\rho \circ \pi + \Psi) = -k(\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi),$$

on $B_o(\frac{R}{2}) \times D$. So for $k \geq 1$, we have

$$\begin{aligned}
(3.7) \quad T_{A,k}^j &= \int_{U_{p_j}} \zeta_j \tilde{d}\tilde{d}^c \Psi_{A,k} \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta \\
&= \int_{\Omega_{p_j}^{-1}([-\frac{R}{2}, \frac{R}{2}] \times B_o(\frac{R}{2}))} \zeta_j \tilde{d}\tilde{d}^c \Psi_{A,k} \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta \\
&= \int_{[-\frac{R}{2}, \frac{R}{2}] \times B_o(\frac{R}{2})} (\zeta_j \circ \Omega_{p_j}^{-1}) \Omega_{p_j}^{-1*} (\tilde{d}\tilde{d}^c \Psi_{A,k} \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n \wedge \eta) \\
&= \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{B_o(\frac{R}{2})} (\zeta_j \circ \Omega_{p_j}^{-1}) \Omega_{p_j}^{-1*} (\tilde{d}\tilde{d}^c \Psi_{A,k} \wedge (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^n) \wedge dx \\
&\geq - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{B_o(\frac{R}{2})} k(\zeta_j \circ \Omega_{p_j}^{-1}) \Omega_{p_j}^{-1*} ((\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^{n+1}) \wedge dx \\
&= -k \int_{U_{p_j}} \zeta_j (\pi^* d\eta + \tilde{d}\tilde{d}^c \Psi)^{n+1} \wedge \eta \\
&= 0,
\end{aligned}$$

where we use the geodesic equation in the last equality and the property

$$dd^c \max\{u, v\} \geq \max\{dd^c u, dd^c v\}$$

as a current for two psh functions u and v in the inequality.

Hence, invoking the dominated convergence theorem gives the following local weak convergence

$$(3.8) \quad \lim_{k \rightarrow \infty} T_{A,k}^j = T_A^j.$$

In particular, $T_A^j \geq 0$, so is T_A which concludes the proof of the theorem. \square

Similar to the argument in [1], we can also prove that $\mathcal{M}(\varphi)$ is continuous along the weak geodesics, and hence is convex.

Theorem 3.4. *\mathcal{M} is continuous and convex along the weak geodesics given in Lemma 2.5.*

Proof. Let $\{\zeta_j\}$ be the partition of unity as in the proof of Theorem 3.3, and $\kappa_\varepsilon(s)$ be a sequence of strictly convex functions with $\kappa'_\varepsilon \geq 1$ on $\{s | s \leq C\}$ tending to s as $\varepsilon \rightarrow 0$, where C is the upper bound of $f_{A,k}(\tau) = \log \Psi_{A,k} \frac{(d\eta_{\varphi_\tau})^n \wedge \eta}{(d\eta)^n \wedge \eta}$, for $\Psi_{A,k}$ and Ψ in Theorem 3.3. In particular, the functions $\kappa_\varepsilon(s)$ can be defined by

$$(3.9) \quad \kappa_\varepsilon(s) = -\sqrt{(s - C - 2\sqrt{\varepsilon})^2 - \varepsilon} + C.$$

With the notations in the proof of Theorem 3.3, we denote

$$(3.10) \quad H_{\varepsilon,A,k}^j(\tau) = \int_M \zeta_j \kappa_\varepsilon(f_{A,k}(\tau)) (d\eta)^n \wedge \eta,$$

$$(3.11) \quad \mathcal{E}_j(\varphi) = \sum_{i=0}^n \int_M \zeta_j \varphi (d\eta_\varphi)^{n-i} \wedge (d\eta)^i \wedge \eta,$$

$$(3.12) \quad \mathcal{E}_j^T(\varphi) = \sum_{i=0}^{n-1} \int_M \zeta_j \varphi (d\eta_\varphi)^{n-i-1} \wedge (d\eta)^i \wedge T \wedge \eta.$$

Furthermore, let $\mathcal{M}_\varepsilon^{\Psi_{A,k}} = \sum_j \mathcal{M}_{\varepsilon,j}^{\Psi_{A,k}}$, where

$$(3.13) \quad \mathcal{M}_{\varepsilon,j}^{\Psi_{A,k}} = \frac{\bar{S}^T}{n+1} \mathcal{E}_j - 2\mathcal{E}_j^{\rho_{d\eta}^T} + 2H_{\varepsilon,A,k}^j.$$

Similar to the proof of Theorem 3.3, we consider $d_\tau d_\tau^c \mathcal{M}_{\varepsilon,j}^{\Psi_{A,k}}$. Indeed, with the notation $T_{\varepsilon,A,k}^j = \int_M \zeta_j \kappa'_\varepsilon \tilde{d} \tilde{d}^c \Psi_{A,k} \wedge (\pi^* d\eta + \tilde{d} \tilde{d}^c \Psi)^n \wedge \eta \geq 0$, we have that

$$(3.14) \quad \begin{aligned} d_\tau d_\tau^c \mathcal{M}_{\varepsilon,j}^{\Psi_{A,k}} &= -2 \int_M \zeta_j (1 - \kappa'_\varepsilon) (\pi^* d\eta + \tilde{d} \tilde{d}^c \Psi)^n \wedge \pi^* \rho_{d\eta}^T \wedge \eta \\ &\quad + 2 \int_M \zeta_j \kappa''_\varepsilon d_\tau f \wedge d_\tau^c f \wedge (d\eta)^n \wedge \eta + T_{\varepsilon,A,k}^j \\ &\geq -C_0 \end{aligned}$$

where we use the convexity of κ_ε and the fact that $\kappa'_\varepsilon \in [1, \frac{2}{\sqrt{3}}]$. We know that $\mathcal{M}_{\varepsilon,j}^{\Psi_{A,k}} + C_0 t^2$ is convex, since it is weak convex and the local Bergman kernels depend continuously on τ . Let k tend to ∞ , we know that $\mathcal{M}_{\varepsilon,j}^{\Psi_A} + C_0 t^2$ is convex, where $\mathcal{M}_{\varepsilon,j}^{\Psi_A}$ is the functional replacing $\Psi_{A,k}$ by Ψ_A in $\mathcal{M}_{\varepsilon,j}^{\Psi_{A,k}}$.

If we sum over j , we know that $\mathcal{M}_\varepsilon^{\Psi_A} + \tilde{C} t^2 = \sum_j \mathcal{M}_{\varepsilon,j}^{\Psi_A} + \tilde{C} t^2$ is also convex. We conclude that $\mathcal{M}^{\Psi_A} + \tilde{C} t^2$ is also convex by $\varepsilon \rightarrow 0$. So \mathcal{M}^{Ψ_A} is continuous on $(0, 1)$ and upper semi-continuous on $[0, 1]$. In particular, \mathcal{M}^{Ψ_A} is convex, since we have known that it is weakly convex. Let $A \rightarrow \infty$, we know that \mathcal{M} is convex, which means that \mathcal{M} is continuous on $(0, 1)$ and upper semi-continuous on $[0, 1]$.

In order to complete the proof of this theorem, we just prove that \mathcal{M} is continuous. Indeed, $(d\eta_{\varphi_t})^n \wedge \eta$ is continuous in the weak $*$ -topology, if φ_t is the geodesic with weak C^2 regularity. Combining the continuity of \mathcal{E} and \mathcal{E}^T and the entropy part is lower semi-continuous in the weak $*$ -topology of measure (see Proposition 3.1), we know that \mathcal{M} is also continuous. \square

Lemma 3.5. *Given u_0, u_1 in \mathcal{H} , let u_t be the corresponding weak geodesic. Then*

$$(3.15) \quad \lim_{t \rightarrow 0^+} \frac{\mathcal{M}(u_t) - \mathcal{M}(u_0)}{t} \geq \int_M \frac{du_t}{dt} \Big|_{t=0^+} (\bar{S}^T - S_{u_0}^T) (d\eta_{u_0})^n \wedge \eta.$$

Proof. We first consider the entropy part H_{μ_0} of \mathcal{M} . According to the convexity of H_{μ_0} with respect to the affine structure of probability measures, we know that

$$(3.16) \quad H_{\mu_0}(\nu_1) - H_{\mu_0}(\nu_0) \geq \left. \frac{dH_{\mu_0}(\nu_s)}{ds} \right|_{s=0},$$

where $\nu_s = s\nu_1 + (1-s)\nu_0$. The monotone convergence implies that

$$\left. \frac{dH_{\mu_0}(\nu_s)}{ds} \right|_{s=0} = \int_M \log \frac{\nu_0}{\mu_0} (\nu_1 - \nu_0).$$

In particular, $\nu_1 = (d\eta_{u_t})^n \wedge \eta$ and $\nu_0 = (d\eta_{u_0})^n \wedge \eta$, we have that

$$\begin{aligned}
(3.17) \quad & \frac{1}{t} (H_{\mu_0}((d\eta_{u_t})^n \wedge \eta) - H_{\mu_0}((d\eta_{u_0})^n \wedge \eta)) \\
& \geq \int_M \log \frac{(d\eta_{u_0})^n \wedge \eta}{\mu_0} \frac{1}{t} ((d\eta_{u_t})^n \wedge \eta - (d\eta_{u_0})^n \wedge \eta) \\
& = \int_M \frac{u_t - u_0}{t} d_B d_B^c \left(\log \frac{(d\eta_{u_0})^n \wedge \eta}{\mu_0} \right) \wedge \left(\sum_{j=0}^{n-1} (d\eta_{u_t})^{n-j-1} \wedge (d\eta_{u_0})^j \right) \wedge \eta.
\end{aligned}$$

We get the estimate of the entropy part by $t \rightarrow 0$, i.e.

$$\begin{aligned}
(3.18) \quad & \lim_{t \rightarrow 0^+} \frac{1}{t} (H_{\mu_0}((d\eta_{u_t})^n \wedge \eta) - H_{\mu_0}((d\eta_{u_0})^n \wedge \eta)) \\
& \geq n \int_M \frac{du_t}{dt} \Big|_{t=0^+} (\rho_{d\eta}^T - \rho_{d\eta_{u_0}}^T) \wedge (d\eta_{u_0})^n \wedge \eta.
\end{aligned}$$

From the equation (3.1), we know that

$$(3.19) \quad \lim_{t \rightarrow 0^+} \frac{\bar{S}^T}{n+1} \frac{\mathcal{E}(u_t) - \mathcal{E}(u_0)}{t} = \bar{S}^T \int_M \frac{du_t}{dt} \Big|_{t=0^+} (d\eta_{u_0})^n \wedge \eta$$

and

$$(3.20) \quad \lim_{t \rightarrow 0^+} \frac{\mathcal{E}^{\rho_{d\eta}^T}(u_t) - \mathcal{E}^{\rho_{d\eta}^T}(u_0)}{t} = n \int_M \frac{du_t}{dt} \Big|_{t=0^+} (d\eta_{u_0})^{n-1} \wedge \rho_{d\eta}^T \wedge \eta.$$

According to the Proposition 3.2, we get the required result. \square

A direct consequence of Theorem 3.4 and the lemma above is the following corollary:

Corollary 3.6. *If $u_0 \in \mathcal{H}$ is a cscS metric, then u_0 is a minimum of \mathcal{M} in \mathcal{H} .*

Proof. If the result is not true, we assume that there exists $u_1 \in \mathcal{H}$ satisfies that $\mathcal{M}(u_1) < \mathcal{M}(u_0)$. Let $\{u_t\}_{t=0}^1$ is the corresponding weak geodesic connecting u_0 and u_1 . According to the convexity of \mathcal{M} with respect to t , we have that

$$(3.21) \quad \mathcal{M}(u_t) \leq t\mathcal{M}(u_1) + (1-t)\mathcal{M}(u_0),$$

which is equivalent to

$$(3.22) \quad \frac{\mathcal{M}(u_t) - \mathcal{M}(u_0)}{t} \leq \mathcal{M}(u_1) - \mathcal{M}(u_0).$$

Hence $\lim_{t \rightarrow 0^+} \frac{\mathcal{M}(u_t) - \mathcal{M}(u_0)}{t} \leq \mathcal{M}(u_1) - \mathcal{M}(u_0)$. However, according to u_0 is a cscS metric and Lemma 3.5, we know that

$$(3.23) \quad \lim_{t \rightarrow 0^+} \frac{\mathcal{M}(u_t) - \mathcal{M}(u_0)}{t} \geq 0,$$

which is a contradiction with $\mathcal{M}(u_1) < \mathcal{M}(u_0)$. \square

4. UNIQUENESS OF CSCS METRICS

In this section, We consider the uniqueness of cscS metrics in the space \mathcal{H} modulo the action generated by the Hamiltonian transverse holomorphic vector fields.

Definition 4.1 ([7]). Fixed a transverse holomorphic structure $(\nu(\mathcal{F}_\xi), \bar{J})$ on the characteristic foliation \mathcal{F}_ξ . A complex vector field X on M is called a transverse holomorphic vector field if it satisfies:

- (1) $\pi([\xi, X]) = 0$;
- (2) $\bar{J}(\pi(X)) = \sqrt{-1}\pi(X)$;
- (3) $\pi([Y, X]) - \sqrt{-1}\bar{J}\pi([Y, X]) = 0, \forall Y$ satisfying $\bar{J}\pi(Y) = -\sqrt{-1}\pi(Y)$,

where π is the the projection to $\nu(\mathcal{F}_\xi)$.

Let $h^T(\xi, \bar{J})$ denote the set of all transverse holomorphic vector fields. One can easily check that $h^T(\xi, \bar{J})$ is a Lie algebra. Let X be a transverse holomorphic vector fields and f be a real-valued function, then $X + f\xi$ is also a transverse holomorphic vector fields. So, $h^T(\xi, \bar{J})$ cannot have finite dimension. But, by [7], we know that $h^T(\xi, \bar{J})/L_\xi$ has finite dimension.

Definition 4.2 ([9]). Let (M, ξ, η, Φ, g) be a compact Sasakian manifold. The automorphism group G of the transverse holomorphic structure is the group of all biholomorphic automorphisms of $(C(M), J)$ which commute with the holomorphic flow generated by $\xi - \sqrt{-1}J\xi$. Its identity component will be denoted by G_0 .

It is well known ([9]) that G_0 acts on the space of all Sasakian metrics on M which is compatible with g .

Definition 4.3 ([10]). A complex vector field X on (M, ξ, η, Φ, g) is called a Hamiltonian transverse holomorphic vector field if it is transverse holomorphic and the complex valued basic function $\psi_X = \sqrt{-1}\eta(X)$ satisfies:

$$(4.1) \quad \bar{\partial}_B \psi_X = -\frac{\sqrt{-1}}{2} d\eta(X, \cdot).$$

Let $\mathfrak{g}^T(\xi, \bar{J})$ denote the set of all Hamiltonian transverse holomorphic vector fields, it is easy to see that $\mathfrak{g}^T(\xi, \bar{J})$ is a Lie algebra.

Proposition 4.1 ([9]). *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold. Then the Lie algebra of the automorphism group G of transverse holomorphic structure is the Lie algebra $\mathfrak{g}^T(\xi, \bar{J})$ of all Hamiltonian transverse holomorphic vector fields.*

Furthermore, we can conclude that $\mathfrak{g}^T(\xi, \bar{J})$ is of finite dimension since the dimension of $h^T(\xi, \bar{J})/L_\xi$ is finite. Indeed, if X_1, X_2 are two different Hamiltonian transverse holomorphic vector fields but $X_1 = X_2 + f\xi$ for some basic function f , then by the definition of Hamiltonian transverse holomorphic vector field, we have that

$$(4.2) \quad \bar{\partial}_B(\sqrt{-1}\eta(X_2 + f\xi)) = -\frac{\sqrt{-1}}{2} d\eta(X_2 + f\xi, \cdot),$$

or equivalently,

$$(4.3) \quad \bar{\partial}_B(\sqrt{-1}\eta(X_2) + \sqrt{-1}f) = -\frac{\sqrt{-1}}{2} d\eta(X_2, \cdot).$$

Hence $\bar{\partial}_B f = 0$, i.e. $f \equiv \text{Constant}$, which implies that

$$(4.4) \quad \mathbf{g}^T(\xi, \bar{J}) \subset h^T(\xi, \bar{J})/L_\xi \oplus \mathbb{C}\xi,$$

and $\mathbf{g}^T(\xi, \bar{J})$ is of finite dimension.

Let φ be a complex valued basic function, the Hamiltonian vector field $\partial_{d\eta}^\# \varphi$ of φ corresponding to the transverse Kähler form $d\eta$ is defined by:

$$\begin{aligned} (1) \quad & \bar{J}(\pi(\partial_{d\eta}^\# \varphi)) = \sqrt{-1}\pi(\partial_{d\eta}^\# \varphi), \\ (2) \quad & \varphi = \sqrt{-1}\eta(\partial_{d\eta}^\# \varphi), \\ (3) \quad & \bar{\partial}_B \varphi(\cdot) = -\frac{\sqrt{-1}}{2}d\eta(\partial_{d\eta}^\# \varphi, \cdot). \end{aligned}$$

As in [7], we denote the Lichnerowicz operator $L_{d\eta}^B$ as follow:

$$(4.5) \quad L_{d\eta}^B \varphi := \frac{1}{4}(\Delta_B^2 \varphi + 4(\rho^T, \sqrt{-1}\partial_B \bar{\partial}_B \varphi) + 2i_{\partial_{d\eta}^\# \varphi} \partial S^T).$$

According to [7, 10], the kernel $\mathcal{H}_{d\eta}^B$ of $L_{d\eta}^B$ is just all the basic functions $\varphi \in \mathcal{H}$ such that $\partial_{d\eta}^\# \varphi$ is transverse holomorphic.

Now we prove the uniqueness of cscS metrics. It should be noted that if $\mathcal{H}_{d\eta}^B$ is trivial, i.e. there is no Hamiltonian transverse holomorphic vector fields over M , Guan and the second author proved that such metric is unique in [14]. And we will use the method of perturbation in [1] to prove the uniqueness of cscS metrics while $\mathcal{H}_{d\eta}^B$ is non-trivial.

Let $\mu > 0$ be a basic smooth volume form on M with the following normalization

$$(4.6) \quad \int_M \mu = \int_M (d\eta)^n \wedge \eta.$$

Similar to [1], we define the function

$$(4.7) \quad \tilde{\mathcal{F}}_\mu(u) = \int_M u \mu - \frac{\mathcal{E}(u)}{n+1} := I_\mu(u) - \frac{\mathcal{E}(u)}{n+1}$$

where \mathcal{E} is the energy functional in section 3. The differential of $\tilde{\mathcal{F}}_\mu$ at $u \in \mathcal{H}$ is

$$(4.8) \quad d\tilde{\mathcal{F}}_\mu \Big|_u = \mu - (d\eta_u)^n \wedge \eta.$$

For the functional I_μ , we have the following inequality as in [1].

Proposition 4.2. *I_μ is strictly convex along weak C^2 geodesic $\{\varphi_t\}$, in the sense that if $f(t) := I_\mu(\varphi_t)$ is affine, then for any t , $d\eta_{\varphi_t} = d\eta_{\varphi_0}$. More precisely, if $d\eta_{\varphi_t} = d\eta + dd^c \varphi_t \leq C d\eta$ and $\mu \geq A(d\eta)^n \wedge \eta$, then*

$$(4.9) \quad f'(1) - f'(0) \geq \frac{\delta A}{C^{n+1}} d(d\eta_{\varphi_0}, d\eta_{\varphi_1})^2,$$

where $\delta > 0$ only depends on μ , η and M , and $d(d\eta_{\varphi_0}, d\eta_{\varphi_1})$ is the distance between $d\eta_{\varphi_0}$ and $d\eta_{\varphi_1}$ defined in [14].

Proof. The C^1 -regularity of φ_t implies that $f'(t) = \int_M \dot{\varphi}_t \mu$ is continuous. In order to get the estimate (4.9), we will apply the result of [14], that is we can approximate $\{\varphi_t\}$, by a smooth sequence $\{\varphi_{t,\varepsilon}\}$, and $\ddot{\varphi}_{t,\varepsilon} - |\partial_B \dot{\varphi}_{t,\varepsilon}|_{d\eta_{\varphi_{t,\varepsilon}}}^2 \geq 0$. Furthermore, we know that the constant C is still valid, since $\varphi_{t,\varepsilon}$ and $\dot{\varphi}_{t,\varepsilon}$ converges to φ_t .

With the notation above, we can compute directly

$$\begin{aligned}
(4.10) \quad \frac{d^2}{dt^2} I_\mu(\varphi_{t,\varepsilon}) &= \int_M \ddot{\varphi}_{t,\varepsilon} \mu \\
&\geq \int_M |\partial_B \dot{\varphi}_{t,\varepsilon}|_{d\eta_{\varphi_{t,\varepsilon}}}^2 \mu \\
&= C^{-1} \int_M |\partial_B \dot{\varphi}_{t,\varepsilon}|_{d\eta}^2 \mu \\
&\geq \frac{\delta}{C} \int_M |\dot{\varphi}_{t,\varepsilon} - a_{t,\varepsilon}|^2 \mu,
\end{aligned}$$

where $a_{t,\varepsilon}$ is the average of $\dot{\varphi}_{t,\varepsilon}$ under the measure μ .

Integrating from 0 to 1, we get that

$$(4.11) \quad \left. \frac{dI_\mu(\varphi_{t,\varepsilon})}{dt} \right|_{t=1} - \left. \frac{dI_\mu(\varphi_{t,\varepsilon})}{dt} \right|_{t=0} \geq \frac{\delta}{C} \int_0^1 \int_M |\dot{\varphi}_{t,\varepsilon} - a_{t,\varepsilon}|^2 \mu dt.$$

The continuity of the differential of I_μ implies that

$$(4.12) \quad f'(1) - f'(0) \geq \frac{\delta}{C} \int_0^1 \int_M |\dot{\varphi}_t - a_t|^2 \mu dt,$$

where a_t is the average of $\dot{\varphi}_t$ under the measure μ . Hence if f is affine, we have $\dot{\varphi}_t = a_t$, i.e. $d\eta_{\varphi_t} = d\eta_{\varphi_0}$.

For the last statement, we argue as following

$$\begin{aligned}
f'(1) - f'(0) &\geq \frac{\delta}{C} \int_0^1 \int_M |\dot{\varphi}_t - a_t|^2 \mu dt \\
&\geq \frac{A\delta}{C} \int_0^1 \int_M |\dot{\varphi}_t - a_t|^2 (d\eta)^n \wedge \eta \wedge dt \\
&\geq \frac{A\delta}{C^{n+1}} \int_0^1 \int_M |\dot{\varphi}_t - a_t|^2 (d\eta_{\varphi_t})^n \wedge \eta \wedge dt \\
&\geq \frac{A\delta}{C^{n+1}} d(d\eta_{\varphi_0}, d\eta_{\varphi_1})^2. \quad \square
\end{aligned}$$

By the result of [15], we know that for any basic smooth volume form μ on M , there exists a basic function $u \in \mathcal{H}$ such that

$$(4.13) \quad (d\eta + d_B d_B^c u)^n \wedge \eta = \mu.$$

Hence we can get the following lemma for the functional $\tilde{\mathcal{F}}_\mu$:

Lemma 4.3. *Let μ and ν be two smooth basic volume forms with total mass equal to $\int_M (d\eta)^n \wedge \eta$. Then for all $\varphi \in \mathcal{H}$:*

$$(4.14) \quad |\tilde{\mathcal{F}}_\mu(\varphi) - \tilde{\mathcal{F}}_\nu(\varphi)| \leq C_{\mu,\nu}.$$

Proof. Assume $\mu = (d\eta_{u_\mu})^n \wedge \eta$ and $\nu = (d\eta_{u_\nu})^n \wedge \eta$. Then

$$\begin{aligned}
&\left| \tilde{\mathcal{F}}_\mu(\varphi) - \tilde{\mathcal{F}}_\nu(\varphi) \right| \\
&= |I_\mu(\varphi) - I_\nu(\varphi)| \\
&= \left| \int_M \varphi d_B d_B^c (u_\mu - u_\nu) \left(\sum_{i=0}^{n-1} (d_B d_B^c u_\mu)^i \wedge (d_B d_B^c u_\mu)^{n-i-1} \right) \wedge \eta \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \int_M (u_\mu - u_\nu) d_B d_B^c \varphi \left(\sum_{i=0}^{n-1} (d_B d_B^c u_\mu)^i \wedge (d_B d_B^c u_\mu)^{n-i-1} \right) \wedge \eta \right| \\
&= \left| \int_M (u_\mu - u_\nu) (d\eta_\varphi - d\eta) \wedge \left(\sum_{i=0}^{n-1} (d_B d_B^c u_\mu)^i \wedge (d_B d_B^c u_\mu)^{n-i-1} \right) \wedge \eta \right| \\
&\leq OSC(u_\mu - u_\nu) = C_{\mu, \nu}. \quad \square
\end{aligned}$$

We denote $F(u) = d\mathcal{M}|_u$ to be the differential of \mathcal{M} at u . Here, we regard $F(u)$ (or F if it is clear) as a 1-form on \mathcal{H} , i.e. for v in the tangent space of \mathcal{H} at u , $F(u)$ acts on v by

$$(4.15) \quad F(u) \cdot v = \int_M v(\bar{S}^T - S_u^T)(d\eta)^n \wedge \eta.$$

According to the computation of [14], the Hessian of \mathcal{M} at $u \in \mathcal{H}$ is equal to

$$\begin{aligned}
(4.16) \quad (Hess\mathcal{M})_u(\psi_0, \psi_1) &= (Dd\mathcal{M})(\psi_0, \psi_1) = (DF)(\psi_0, \psi_1) \\
&= \frac{1}{2} \int_M \psi_0 \mathfrak{D}_{d\eta_u}^* \mathfrak{D}_{d\eta_u} \psi_1 (d\eta_u)^n \wedge \eta \\
&= \frac{1}{2} \int_M \langle \mathfrak{D}_{d\eta_u} \psi_0, \mathfrak{D}_{d\eta_u} \psi_1 \rangle_{d\eta_u} (d\eta_u)^n \wedge \eta,
\end{aligned}$$

where D is the connection defined in (2.5) and $\mathfrak{D}_{d\eta_u} = \bar{\partial}_B \partial_{d\eta_u}^\#$.

For a basic smooth volume form ν on M , we define a functional G_ν on \mathcal{H} by

$$(4.17) \quad G_\nu \cdot w = \int_M w \nu.$$

and consider the solution v to the equation

$$(4.18) \quad D_v F|_u = G_\nu.$$

Similarly to [1], we know that (4.18) is solvable if and only if $G_\nu \cdot w = 0$ for all $w \in \mathcal{H}_{d\eta_u}^B$, i.e. all basic w such that $\partial_{d\eta_u}^\# w$ is transverse holomorphic.

Let u_0, u_1 be two different¹ smooth basic functions with cscS metrics such that $\mathcal{I}(u_0) = \mathcal{I}(u_1) = 0$. Modifying the argument of [1] for Kähler case, we show that after a preliminary modification of $u_i (i = 0, 1)$ by applying an action generated by $\mathbf{g}^T(\xi, \bar{J})$, the equation

$$(4.19) \quad D_{v_i} F|_{u_i} = -G_{\nu_i}$$

is solvable, where G_{ν_i} is the differential of $\tilde{\mathcal{F}}_\mu$ in (4.8) at u_i ². For the convenience of the readers, we will give details of the proof here.

Lemma 4.4. *If V is a Hamiltonian transverse holomorphic vector field, then it determines a geodesic ray in \mathcal{K} following the flow of V .*

Proof. We denote $d\eta_{\tilde{u}_t} = d\eta + d_B d_B^c u_t = \exp(tV)^* d\eta_{u_0}$, where $\tilde{u}_t \in \mathcal{H}$ is smooth with respect to t and $\tilde{u}_0 = u_0$. Since V is a Hamiltonian transverse holomorphic

¹Saying u_0 and u_1 are different, we mean $u_0 - u_1$ is not transverse constant, i.e. $d\eta_{u_0}$ and $d\eta_{u_1}$ are two different transverse Kähler metrics.

²We will also use the notation u_i for the function after action.

vector field, we know that there exists $h_{\tilde{u}_t}^V \in \mathcal{H}$ such that $V = \partial_{d\eta_{\tilde{u}_t}}^\# h_{\tilde{u}_t}^V$ and $h_{\tilde{u}_t}^V = h_{\tilde{u}_0}^V + V(\tilde{u}_t - \tilde{u}_0)$. Hence we have that

$$\begin{aligned}
L_V d\eta_{\tilde{u}_t} &= \frac{d}{dt} \exp(tV)^* d\eta_{u_0} \\
(4.20) \quad &= \sqrt{-1} d_B d_B^c \dot{\tilde{u}}_t \\
&= \sqrt{-1} d_B d_B^c h_{\tilde{u}_t}^V,
\end{aligned}$$

which implies that $\dot{\tilde{u}}_t = h_{\tilde{u}_t}^V + f(t)$ for $f \in C^\infty(\mathbb{R})$. Let $\hat{u}_t = \tilde{u}_t - \int_0^t f dt$, then we can conclude that $\dot{\hat{u}}_t = h_{\hat{u}_t}^V$ and $V = \partial_{d\eta_{\hat{u}_t}}^\# \dot{\hat{u}}_t$. Furthermore,

$$\begin{aligned}
&\frac{d}{dt} \int_M \dot{\hat{u}}_t (d\eta_{\hat{u}_t})^n \wedge \eta \\
(4.21) \quad &= \frac{d}{dt} \int_M h_{\hat{u}_t}^V (d\eta_{\hat{u}_t})^n \wedge \eta \\
&= \int_M V(\dot{\hat{u}}_t) (d\eta_{\hat{u}_t})^n \wedge \eta + \int_M h_{\hat{u}_t}^V \triangle_B \dot{\hat{u}}_t (d\eta_{\hat{u}_t})^n \wedge \eta \\
&= \int_M V(\dot{\hat{u}}_t) (d\eta_{\hat{u}_t})^n \wedge \eta - \int_M \langle \partial_{d\eta_{\hat{u}_t}}^\# h_{\hat{u}_t}^V, \nabla_B \dot{\hat{u}}_t \rangle_{d\eta_{\hat{u}_t}} (d\eta_{\hat{u}_t})^n \wedge \eta \\
&= \int_M V(\dot{\hat{u}}_t) (d\eta_{\hat{u}_t})^n \wedge \eta - \int_M V(\dot{\hat{u}}_t) (d\eta_{\hat{u}_t})^n \wedge \eta \\
&= 0,
\end{aligned}$$

so we have that $\int_M \dot{\hat{u}}_t (d\eta_{\hat{u}_t})^n \wedge \eta = C$. If we denote $u_t = \hat{u}_t - Ct$, then we can conclude that $\mathcal{I}(u_t) \equiv 0$, i.e. $u_t \in \mathcal{H}_0$, since $\mathcal{I}(C_t) = C_t$ where C_t is some constant of t . So we have that

$$\begin{aligned}
\ddot{u}_t &= \dot{h}_{\hat{u}_t}^V = V(\dot{\hat{u}}_t) = V(\dot{u}_t) \\
(4.22) \quad &= d\eta_{\tilde{u}_t}(\partial_{d\eta_{\tilde{u}_t}}^\# \dot{\hat{u}}_t, \nabla^B \dot{\hat{u}}_t) = \left\| \partial_B \dot{\hat{u}}_t \right\|_{d\eta_{\tilde{u}_t}} = \left\| \partial_B \dot{u}_t \right\|_{d\eta_{u_t}}.
\end{aligned}$$

Hence u_t is a geodesic, and so is $d\eta_{u_t}$, i.e. the ray determined by V . □

Proposition 4.5. *Let S be the submanifold of \mathcal{H}_0 consisting of all potentials of metrics $\iota^* d\eta_{u_i}$ ($i = 0, 1$), where ι ranges over the actions generated by $\mathbf{g}^T(\xi, \bar{J})$. Then $\tilde{\mathcal{F}}_\mu$ has a minimum and hence a critical point on S . This implies that G_{ν_i} annihilates all basic functions w such that $\partial_{d\eta_{u_i}}^\# w$ is transverse holomorphic.*

Proof. According to Lemma 4.4, any Hamiltonian transverse holomorphic vector field V determines a geodesic ray starting at u_i . And S is the union of all such rays. If $\mu = (d\eta_{u_i})^n \wedge \eta$, then u_i is a critical point of $\tilde{\mathcal{F}}_\mu$. Since $\tilde{\mathcal{F}}_\mu$ is strictly convex along each ray, it follows that $\tilde{\mathcal{F}}_\mu$ is proper on each ray. And here we say a function $f(t)$ is proper if and only if $\lim_{t \rightarrow +\infty} f(t) = +\infty$. Since the dimension of $\mathbf{g}^T(\xi, \bar{J})$ is finite, $\tilde{\mathcal{F}}_\mu$ is proper on S in this case. Lemma 4.3 implies that $\tilde{\mathcal{F}}_\mu$ is proper on S for any choice of μ . Hence it has a minimum on S . And for convenience, we still use the notation u_i for the function which achieves the minimum of $\tilde{\mathcal{F}}_\mu$ on S .

Let $\iota_t = \exp(tV)$ be the one-parameter group of transformations determined by a Hamiltonian transverse holomorphic vector field $V = \partial_{d\eta_{u_i}}^\# h_{u_i}^V$. For $u_{i,t} \in \mathcal{H}_0$

which is the potential of $d\eta_{u_i,t} = \iota_t^* d\eta_{u_i}$, we have

$$(4.23) \quad L_V d\eta_{u_i} = \sqrt{-1} \partial_B \bar{\partial}_B h_{u_i}^V,$$

and

$$(4.24) \quad \frac{d(g_t^* d\eta_{u_i})}{dt} = \frac{d(d\eta + \sqrt{-1} \partial_B \bar{\partial}_B u_{i,t})}{dt} = \sqrt{-1} \partial_B \bar{\partial}_B \dot{u}_{i,t}.$$

Hence $h_{u_i}^V = \dot{u}_{i,t}|_{t=0} + C$ according to the argument of Lemma 4.4. We have that

$$(4.25) \quad d\tilde{\mathcal{F}}_\mu \cdot h_{u_i}^V = d\tilde{\mathcal{F}}_\mu \cdot \dot{u}_{i,t}|_{t=0} = 0,$$

since u_i is a minimum of $\tilde{\mathcal{F}}_\mu$. \square

Remark 4.1. After the modification, u_i is still a critical point of \mathcal{M} , since \mathcal{M} is invariant along the flow generated by Hamiltonian transverse holomorphic vector field.

Now, we give a proof of the main theorem.

A proof of Theorem 1.1. Let g_0 and g_1 are two cscS metrics compatible with (M, ξ, η, Φ, g) , and u_0 and u_1 are the transverse Kähler potential functions with respect to g_0 and g_1 . By Proposition 4.5., modulo automorphisms, we can suppose that $u_i (i = 0, 1)$ satisfies that there exists v_i such that

$$(4.26) \quad D_{v_i} F|_{u_i} = -G_{v_i}.$$

And now we prove that $d\eta_{u_0} = d\eta_{u_1}$.

Consider the functional $\mathcal{M}_s = \mathcal{M} + s\tilde{\mathcal{F}}_\mu$. Its differential is

$$(4.27) \quad F_s(u_i) = F(u_i) + s d\tilde{\mathcal{F}}_\mu(u_i) = F(u_i) + s G_{v_i}.$$

For all w_s considered as a tangent vector field in \mathcal{H} along the curve $u_i + sv_i$ we have that

$$(4.28) \quad \left. \frac{d}{ds} \right|_{s=0} F_s(u_i + sv_i) \cdot w_s = D_{v_i} F|_{u_i} \cdot w_0 + F(u_i) D_{v_i} w_s + G_{v_i} \cdot w_0 = 0.$$

Since $F_s(u_i + sv_i)$ is a linear functional on the the space of basic functions, we have that $|F_s(u_i + sv_i) \cdot w| \leq C \sup_M |w| o(s)$. Indeed, we can write

$$(4.29) \quad F_s(u_i + sv_i) \cdot w = \int_M w f(s, x) dV,$$

for some function f on $\mathbb{R} \times M$. Equation (4.28) implies that $\left. \frac{\partial f(s, x)}{\partial s} \right|_{s=0} = 0$. Furthermore $F_s(u_i + sv_i) \cdot w = 0$ at $s = 0$, arguing by the differential mean value theorem, we have that

$$(4.30) \quad \left| F_s(u_i + sv_i) \cdot \frac{w}{\sup_M |w|} - 0 \right| = |s| \left| \int_M \frac{w}{\sup_M |w|} \frac{\partial f(t, x)}{\partial t} \Big|_{t=\theta(s, x)} dV \right|,$$

where $|\theta(s, x)| < |s|$. Letting $s \rightarrow 0$, we conclude that

$$|F_s(u_i + sv_i) \cdot w| \leq C \sup_M |w| o(s).$$

Lemma 2.5 implies that we can connect $u_0^s = u_0 + sv_0$ and $u_1^s = u_1 + sv_1$ by a unique weak C^2 geodesic u_t^s . In particular, according to the regularity in [14],

we know that $\|u_t^s\|_{C^1(M \times [0,1])}$ and $|\Delta u_t^s|_{M \times [0,1]}$ is bounded for s sufficiently small. According to Lemma 3.5, the convexity of \mathcal{M} implies that

$$(4.31) \quad \frac{d}{dt} \Big|_{t=0+} \mathcal{M}(u_t^s) \geq F(u_0^s) \cdot \frac{du_t^s}{dt} \Big|_{t=0+}$$

and

$$(4.32) \quad \frac{d}{dt} \Big|_{t=1-} \mathcal{M}(u_t^s) \leq F(u_1^s) \cdot \frac{du_t^s}{dt} \Big|_{t=1-}.$$

Since $\tilde{\mathcal{F}}_\mu$ is strictly convex along such weak geodesic, the same inequalities hold for \mathcal{M}_s as well³. The linearity of $\mathcal{E}(u_t^s)$ in t implies that

$$(4.33) \quad \begin{aligned} 0 &\leq s \left(\frac{dI_\mu(u_t^s)}{dt} \Big|_{t=1-} - \frac{dI_\mu(u_t^s)}{dt} \Big|_{t=0+} \right) \\ &\leq \frac{d\mathcal{M}_s(u_t^s)}{dt} \Big|_{t=1-} - \frac{d\mathcal{M}_s}{dt} \Big|_{t=0+} \\ &\leq F_s(u_1^s) \cdot \frac{du_t^s}{dt} \Big|_{t=1-} - F_s(u_0^s) \cdot \frac{du_t^s}{dt} \Big|_{t=0+} \\ &= o(s). \end{aligned}$$

Hence,

$$(4.34) \quad \frac{dI_\mu(u_t^s)}{dt} \Big|_{t=1-} - \frac{dI_\mu(u_t^s)}{dt} \Big|_{t=0+} \leq o(1).$$

According to Proposition 4.2, we know that $d(\eta_{u_0^s}, \eta_{u_1^s}) \leq o(1)$. Hence

$$(4.35) \quad d(\eta_{u_0}, \eta_{u_1}) = 0,$$

which implies that $d\eta_{u_0} = d\eta_{u_1}$. This complete the proof of main theorem. \square

REFERENCES

- [1] R. J. Berman and B. Berndtsson: Convexity of the K -energy on the space of Kähler metrics and uniqueness of extremal metrics. *arXiv:1405.0401*.
- [2] B. Berndtsson: Subharmonicity properties of the bergman kernel and some other functions associated to pseudoconvex domains. *Annales de l'Institut Fourier*, 56(6):1633–1662, (2006).
- [3] Z. Błocki: A gradient estimate in the Calabi–Yau theorem. *Mathematische Annalen*, 344(2):317–327, (2009).
- [4] Z. Błocki: On geodesics in the space of Kähler metrics, Proceedings of the *Conference in Geometry*, dedicated to Shing-Tung Yau (Warsaw, April 2009). *Advances in Geometric Analysis*, ed. S. Janeczko, J. Li, D. Phong, Advanced Lectures in Mathematics 21, pp. 3–20, International Press, (2012).
- [5] C. P. Boyer and K. Galicki: *Sasakian Geometry*. Oxford Univ. Press, (2006).
- [6] C. P. Boyer, K. Galicki and J. Kollár: Einstein metrics on spheres. *Annals of mathematics*, 162:557–580, (2005).
- [7] C. P. Boyer, K. Galicki and S. R. Simanca: Canonical Sasakian metrics. *Communications in Mathematical Physics*, 279(3):705–733, (2008).
- [8] X. X. Chen: The space of Kähler metrics. *Journal of Differential Geometry*, 56(2):189–234, (2000).
- [9] K. Cho, A. Futaki and H. Ono: Uniqueness and examples of compact toric Sasakian-Einstein metrics. *Communications in Mathematical Physics*, 277(2):439–458, (2008).
- [10] A. Futaki, H. Ono, and G. F. Wang: Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds. *Journal of Differential Geometry*, 83(3):585–636, (2009).

³ F on the righthand side of the above inequality will be replaced by F_s as well.

- [11] J.P. Gauntlett, D. Martelli, J. Sparks and W. Waldram: A new infinite class of Sasaki-Einstein manifolds, *Advances in Theoretical and Mathematical Physics*, 8:987-1000, (2004).
- [12] M. Godlinski, W. Kopczynski, and P. Nurowski: Locally Sasakian manifolds. *Classical and Quantum Gravity*, 17:105–115, (2000).
- [13] P. F. Guan and X. Zhang: A Geodesic equation in the space of Sasakian metrics. in: Geometry and Analysis, No. 1, in: Adv. Lectures Math., vol. 17, 2010, pp. 303–318.
- [14] P. F. Guan and X. Zhang: Regularity of the geodesic equation in the space of Sasakian metrics. *Advances in Mathematics*, 230(1):321–371, (2012).
- [15] E. Kacimi-Alaoui: Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications. *Compositio Mathematica*, 73(1):57–106, (1990).
- [16] I. R. Klebanov and E. Witten: Superconformal field theory on threebranes at a Calabi-Yau singularity. *Nuclear Physics B*, 536:199-218, (1999).
- [17] J. Maldacena: The large N limit of superconformal field theories and supergravity. *Advances in Theoretical and Mathematical Physics*, 2:231-252, (1998).
- [18] D. Martelli and J. Sparks: Toric geometry, Sasaki-Einstein manifolds and a new infinite class of ADS/CFT duals. *Communications in Mathematical Physics*, 262(1):51–89, (2006).
- [19] D. Martelli, J. Sparks, and S. T. Yau: Sasaki-Einstein manifolds and volume minimisation. *Communications in Mathematical Physics*, 280(3):611–673, (2008).
- [20] Y. Nitta and K. Sekiya: Uniqueness of Sasakian-Einstein metrics. *Tohoku Mathematical Journal*, 64(3):453-468, (2012).
- [21] S. Sasaki: On differentiable manifolds with certain structures which are closely related to almost-contact structure. *Tohoku Mathematical Journal*, 2:459-476, (1960).
- [22] J. Sparks: Sasaki-Einstein manifolds. *arXiv:1004.2461*.
- [23] X. Zhang: Some invariants in Sasakian geometry. *International Mathematics Research Notices*, 2010.

XISHEN JIN, KEY LABORATORY OF WU WEN-TSUN MATHEMATICS, CHINESE ACADEMY OF SCIENCES, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, 230026, P.R. CHINA,
E-mail address: `jinxsh@mail.ustc.edu.cn`

XI ZHANG, KEY LABORATORY OF WU WEN-TSUN MATHEMATICS, CHINESE ACADEMY OF SCIENCES, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, 230026, P.R. CHINA,
E-mail address: `mathzx@ustc.edu.cn`